

# A Wilsonian Energy-Momentum Tensor

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## Abstract

For local conformal field theories, it is shown how to construct an expression for the energy-momentum tensor in terms of a Wilsonian effective Lagrangian. Tracelessness implies a single, unintegrated equation which enforces both the Exact Renormalization Group equation and its partner encoding invariance under special conformal transformations.

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## I. INTRODUCTION

In the context of quantum field theory, consider the action for a free, scalar field in  $d$ -dimensional Euclidean space:

$$S^{\text{Gauss}}[\varphi] = \frac{1}{2} \int d^d x \, \partial_\mu \varphi \, \partial_\mu \varphi. \quad (1.1)$$

What is the scaling dimension of the field? This is, of course, a trivial question to answer: since the theory is non-interacting, we can just use naïve power counting to obtain  $(d-2)/2$ . However, this method does not carry over to the interacting case; consequently, we seek other, more elaborate ways of answering this question that are more generalizable.

Our next approach—which, at least in its preliminary incarnation, is in a sense just a somewhat more formal means of power-counting—involves constructing a functional representation of the dilatation generator. First define

$$D^{(\delta)} = x \cdot \partial + \delta, \quad (1.2)$$

where  $\delta$  is an *a priori* unknown scaling dimension and, in this context, the dot denotes a contraction of indices. Next introduce

$$\mathcal{D} = D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} \quad (1.3)$$

where here the dot denotes an integral over the position of the field. Demanding dilatation invariance of the action recovers the previous result:

$$D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} S^{\text{Gauss}}[\varphi] = 0 \quad \Rightarrow \quad \delta = \delta_0 \equiv \frac{d-2}{2}. \quad (1.4)$$

As it stands, this method is also of no use in the interacting case, but the approach has an appropriate generalization.

If we wish to retain a representation of the dilatation generator for interacting quantum field theories which is linear in the functional derivative, then we must take it to act on the correlation functions, which are non-local. To preserve a local description, we are forced instead consider representations of the dilatation generator which are (at least) quadratic in derivatives. The Exact Renormalization Group (ERG) provides a particular realization of this:

$$\mathcal{D} = D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} + \varphi \cdot \mathcal{G}^{-1} \cdot G \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi}, \quad (1.5)$$

where  $G$  and  $\mathcal{G}$  incorporate an ultraviolet cutoff function, in a manner to be specified later. Scale invariance is enforced by the fixed-point condition:

$$\mathcal{D} e^{-S} = 0. \quad (1.6)$$

A particularly interesting feature of this equation is that, in general, not only is the scaling dimension *a priori* unknown but, so too, is the action. The correct understanding is to recognize the ERG equation as a non-linear eigenvalue equation [1]. In principle, then, the ERG equation allows the self-consistent determination of the spectrum of local fixed-points. However, the equation itself is fiendishly difficult to attack and, in general, various rather brutal approximation schemes must be employed.

Within the context of approaches that are intrinsically local, are there any options to improve upon the ERG? A clue comes from the fact that (1.6) is only a statement of scale invariance; it does not automatically incorporate full conformal invariance (for a detailed discussion of the relationship between scale and conformal invariance, see [2]). Whilst it is true that for many theories of interest scale invariance in fact enhances to conformal

invariance, this is not a general property of all solutions of (1.6). Indeed, just as (1.5) provides a representation of the dilatation generator, so too is there an associated representation of the special conformal generator [3–5]. The form we use is equivalent to that derived in [4]:

$$\mathcal{K}_\mu = K^{(\delta)}{}_\mu \varphi \cdot \frac{\delta}{\delta \varphi} + \varphi \cdot \mathcal{G}^{-1} \cdot G_\mu \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G_\mu \cdot \frac{\delta}{\delta \varphi} - \eta \partial_\alpha \varphi \cdot K^{-1} \cdot G \cdot \frac{\delta}{\delta \varphi}, \quad (1.7)$$

where  $G_\mu$  is related to  $G$ ,  $K$  is the cutoff function, not to be confused with

$$K^{(\delta)}{}_\mu = 2x_\mu (x \cdot \partial + \delta) - x^2 \partial_\mu, \quad (1.8)$$

and  $\eta$  is the anomalous dimension of  $\varphi$ :

$$\delta = \delta_0 + \eta/2 = \frac{d-2+\eta}{2}. \quad (1.9)$$

A conformally invariant theory thus satisfies not just (1.6) but also

$$\mathcal{K}_\mu e^{-S} = 0. \quad (1.10)$$

However rather than attempting to solve the pair of equations (1.6) and (1.10) as they stand, in this paper we shall instead seek a single equation which incorporates full conformal invariance. To understand how such an equation might arise, let us consider a third answer to the question posed at the start of this paper: we will obtain the scaling dimension of our Gaussian theory via consideration of the energy-momentum tensor. This is significantly more involved than either of the above approaches, but it has the merit of providing an interesting and potentially powerful generalization.

For the Gaussian theory, we can take the energy-momentum tensor to be defined by the following three equations:

$$\partial_\alpha T_{\alpha\beta}^{\text{Gauss}} = -\partial_\beta \varphi \times \frac{\delta S^{\text{Gauss}}}{\delta \varphi}, \quad (1.11a)$$

$$T_{\alpha\beta}^{\text{Gauss}} = T_{\beta\alpha}^{\text{Gauss}}, \quad (1.11b)$$

$$T_{\alpha\alpha}^{\text{Gauss}} = -\delta \varphi \times \frac{\delta S^{\text{Gauss}}}{\delta \varphi}. \quad (1.11c)$$

In position space, the  $\times$  just represents the product of two quantities at the same location; it can be omitted and is generally used to emphasise the lack of an integral (which would be denoted by a dot). These three equations encode, respectively, translational, rotational and scale-invariance of the action; however, these three invariances can only be expressed in

terms of a single object—the energy momentum tensor—if the theory is in fact conformally invariant.

It is instructive to see how these equations can be used to determine  $\delta$ . Substituting (1.1) into (1.11a) and rearranging gives:

$$\partial_\alpha T_{\alpha\beta}^{\text{Gauss}} = \partial_\beta \varphi \times \partial^2 \varphi = \partial_\alpha \left( \partial_\alpha \varphi \times \partial_\beta \varphi - \delta_{\alpha\beta} \frac{1}{2} \partial_\lambda \varphi \times \partial_\lambda \varphi \right). \quad (1.12)$$

Therefore,

$$T_{\alpha\beta}^{\text{Gauss}} = \partial_\alpha \varphi \times \partial_\beta \varphi - \delta_{\alpha\beta} \frac{1}{2} \partial_\lambda \varphi \times \partial_\lambda \varphi + \partial_\lambda W_{\lambda\alpha\beta}, \quad (1.13)$$

where  $W_{\lambda\alpha\beta} = -W_{\alpha\lambda\beta}$  vanishes when contracted with  $\partial_\alpha$ . The condition (1.11b) enforces  $W_{\lambda\alpha\beta} = W_{\lambda\beta\alpha}$ . For the Gaussian theory, where all terms contributing to the energy-momentum tensor must have two derivatives and two powers of the field, it follows that

$$\partial_\lambda W_{\lambda\alpha\beta} = w(\delta_{\alpha\beta} \partial^2 - \partial_\alpha \partial_\beta) \varphi^2, \quad (1.14)$$

where  $w$  is determined by the condition for scale invariance (1.11c):

$$\frac{2-d}{2} \partial_\alpha \varphi \times \partial_\alpha \varphi + w(d-1) \partial^2 \varphi^2 = \delta \varphi \times \partial^2 \varphi. \quad (1.15)$$

A bit of simple algebra reveals that:

$$w = \frac{d-2}{4(d-1)}, \quad \delta = 2w(d-1) = \frac{d-2}{2}, \quad (1.16)$$

as before.

The generalizations of (1.11a), (1.11b) and (1.11c) appropriate to interacting theories within the framework of the ERG were derived in [4] and subsequently explored in [6]. For a putative fixed-point, it is always possible to construct a symmetric, conserved tensor, as argued in [4] and as we shall explicitly see below. However, dilatation invariance is not automatic, and demanding it be satisfied produces a constraint equation, which we shall refer to as the ‘conformal fixed-point equation’. In contrast to the ERG equation (1.6) and its partner (1.10) this equation is unintegrated—reflecting the fact that it involves the Lagrangian rather than just the action. Moreover, this single equation automatically enforces both (1.6) and (1.10), as we shall see. Solutions to the conformal fixed-point equation self consistently determine the action and anomalous dimension of the fundamental field, while simultaneously providing the requisite improvement of the energy-momentum tensor such as to render it traceless. It is beyond the scope of this paper to investigate methods of solving

the conformal fixed-point equation and thus addressing the question of whether or not it confers an advantage over the plain ERG equation and its special conformal partner.

Before diving into the ERG treatment, in section II A we first recall the generalizations of (1.11a), (1.11b) and (1.11c) in an arbitrary representation of the conformal algebra and then describe a strategy for solving these equations. In section II B we apply this method in the context of classical theories, and illustrate it with some concrete examples. The classical solution forms part of the full ERG solution which is presented in section II C. The key results of the rather technical development are summarised in the conclusion; however, it is worth bearing in mind that, having been obtained, the veracity of the conformal fixed-point equation (2.118) can be checked very simply.

## II. THE ENERGY-MOMENTUM TENSOR

### A. General Considerations

As emphasised in [4], if one is to consider various representations of the conformal algebra, then one must be prepared to consider associated representations of the energy-momentum tensor. In this section we keep the representation arbitrary, and to indicate this use the symbol  $\mathcal{T}$ ; as in the introduction, throughout the rest of this paper we work in  $d$ -dimensional Euclidean space. The generalization of (1.11a), (1.11b) and (1.11c) reads [4]:

$$\partial_\alpha \mathcal{T}_{\alpha\beta} = -\hat{\mathcal{O}}^{(d-\delta)} \times \partial_\beta \mathcal{O}^{(\delta)}, \quad (2.1a)$$

$$\mathcal{T}_{\alpha\beta} = \mathcal{T}_{\beta\alpha}, \quad (2.1b)$$

$$\mathcal{T}_{\alpha\alpha} = -\delta \hat{\mathcal{O}}^{(d-\delta)} \times \mathcal{O}^{(\delta)}, \quad (2.1c)$$

where  $\mathcal{O}^{(\delta)}$  is a quasi-primary field of scaling dimension  $\delta$  viz.

$$\mathcal{D}\mathcal{O}^{(\delta)} = D^{(\delta)}\mathcal{O}^{(\delta)}, \quad \mathcal{K}_\mu \mathcal{O}^{(\delta)} = K^{(\delta)}_\mu \mathcal{O}^{(\delta)} \quad (2.2)$$

whereas  $\hat{\mathcal{O}}^{(d-\delta)}$  satisfies

$$[\mathcal{D}, \hat{\mathcal{O}}^{(d-\delta)}] = D^{(\delta)}\hat{\mathcal{O}}^{(d-\delta)}, \quad [\mathcal{K}_\mu, \hat{\mathcal{O}}^{(d-\delta)}] = K^{(\delta)}_\mu \hat{\mathcal{O}}^{(d-\delta)}. \quad (2.3)$$

In the ERG representation,  $\hat{\mathcal{O}}^{(d-\delta)}$  can be constructed from a particular quasi-primary field by adding a functional derivative term, as we will see later.

Suppose that there exists a recipe to extract a conserved (though not necessarily symmetric) contribution to the energy-momentum tensor,  $\overline{\mathcal{T}}_{\alpha\beta}$ . In this case, the solution to (2.1a) can, along the lines of the Belinfante tensor, be expressed as:

$$\mathcal{T}_{\alpha\beta} = \overline{\mathcal{T}}_{\alpha\beta} + \partial_\lambda \mathcal{B}_{\lambda\alpha\beta}, \quad (2.4)$$

where

$$\mathcal{B}_{\lambda\alpha\beta} = -\mathcal{B}_{\alpha\lambda\beta}. \quad (2.5)$$

Integrating (2.4) and using (2.1b) it is apparent that, for local solutions,  $\overline{\mathcal{T}}_{\alpha\beta}$  is symmetric up to total derivatives. Therefore, for some symmetric  $\overline{\mathcal{T}}_{\alpha\beta}^{\text{sym}}$  we may write

$$\overline{\mathcal{T}}_{\alpha\beta} = \overline{\mathcal{T}}_{\alpha\beta}^{\text{sym}} + \partial_\lambda \mathcal{F}_{\lambda\alpha\beta}, \quad (2.6)$$

for some  $\mathcal{F}_{\lambda\alpha\beta}$ . In general,  $\mathcal{F}_{\lambda\alpha\beta}$  may contain pieces both symmetric and anti-symmetric under  $\alpha \leftrightarrow \beta$ . For the analysis in subsequent sections it will turn out to be more convenient to work with  $\mathcal{F}_{\lambda\alpha\beta}$  rather than  $\mathcal{B}_{\lambda\alpha\beta}$  and the following development reflects this. Substituting (2.6) into (2.4) and exploiting (2.1b) yields:

$$\partial_\lambda (\mathcal{F}_{\lambda\alpha\beta} + \mathcal{B}_{\lambda\alpha\beta} - \mathcal{F}_{\lambda\beta\alpha} - \mathcal{B}_{\lambda\beta\alpha}) = 0. \quad (2.7)$$

Next split  $\mathcal{F}_{\lambda\alpha\beta}$  into pieces which are symmetric/antisymmetric under  $\lambda \leftrightarrow \alpha$ :

$$\mathcal{F}_{\lambda\alpha\beta} = \frac{1}{2}(\mathcal{F}_{\lambda\alpha\beta} + \mathcal{F}_{\alpha\lambda\beta}) + \frac{1}{2}(\mathcal{F}_{\lambda\alpha\beta} - \mathcal{F}_{\alpha\lambda\beta}) \quad (2.8)$$

and absorb the antisymmetric piece into  $\mathcal{B}_{\lambda\alpha\beta}$  (which, recall, has the same antisymmetry property):

$$\overline{\mathcal{B}}_{\lambda\alpha\beta} = \mathcal{B}_{\lambda\alpha\beta} + \frac{1}{2}(\mathcal{F}_{\lambda\alpha\beta} - \mathcal{F}_{\alpha\lambda\beta}). \quad (2.9)$$

Substituting into (2.7) gives:

$$\partial_\lambda \left( \frac{1}{2}(\mathcal{F}_{\lambda\alpha\beta} + \mathcal{F}_{\alpha\lambda\beta}) - \overline{\mathcal{B}}_{\lambda\beta\alpha} \right) - \alpha \leftrightarrow \beta = 0. \quad (2.10)$$

Due to the overall antisymmetry of the left-hand side under  $\alpha \leftrightarrow \beta$  we can add zero to the left-hand side in the following way:

$$\partial_\lambda \left( \frac{1}{2}(\mathcal{F}_{\lambda\alpha\beta} + \mathcal{F}_{\alpha\lambda\beta} - \mathcal{F}_{\beta\alpha\lambda} - \mathcal{F}_{\alpha\beta\lambda}) - \overline{\mathcal{B}}_{\lambda\beta\alpha} \right) - \alpha \leftrightarrow \beta = 0. \quad (2.11)$$

The point of doing this is that the  $\mathcal{F}$ -terms are antisymmetric under  $\lambda \leftrightarrow \beta$ . This equation can be solved by taking

$$\bar{\mathcal{B}}_{\lambda\beta\alpha} = \frac{1}{2}(\mathcal{F}_{\lambda\alpha\beta} + \mathcal{F}_{\alpha\lambda\beta} - \mathcal{F}_{\beta\alpha\lambda} - \mathcal{F}_{\alpha\beta\lambda}) + \mathcal{W}_{\lambda\beta\alpha}, \quad (2.12)$$

where

$$\partial_\lambda \mathcal{W}_{\lambda\beta\alpha} = \partial_\lambda \mathcal{W}_{\lambda\alpha\beta}, \quad \mathcal{W}_{\lambda\alpha\beta} = -\mathcal{W}_{\alpha\lambda\beta}. \quad (2.13)$$

Utilizing (2.9) and substituting into (2.4) gives:

$$\mathcal{T}_{\alpha\beta} = \bar{\mathcal{T}}_{\alpha\beta} + \frac{1}{2}\partial_\lambda(\mathcal{F}_{\alpha\lambda\beta} + \mathcal{F}_{\beta\lambda\alpha} + \mathcal{F}_{\lambda\beta\alpha} - \mathcal{F}_{\lambda\alpha\beta} - \mathcal{F}_{\beta\alpha\lambda} - \mathcal{F}_{\alpha\beta\lambda}) + \partial_\lambda \mathcal{W}_{\lambda\alpha\beta}; \quad (2.14)$$

taking the trace yields:

$$\mathcal{T}_{\alpha\alpha} = \bar{\mathcal{T}}_{\alpha\alpha} + \partial_\lambda(\mathcal{F}_{\alpha\lambda\alpha} - \mathcal{F}_{\alpha\alpha\lambda} + \mathcal{W}_{\lambda\alpha\alpha}). \quad (2.15)$$

Comparing this with (2.1c) gives a consistency condition:

$$\bar{\mathcal{T}}_{\alpha\alpha} + \partial_\lambda(\mathcal{F}_{\alpha\lambda\alpha} - \mathcal{F}_{\alpha\alpha\lambda} + \mathcal{W}_{\lambda\alpha\alpha}) = -\delta\hat{\mathcal{O}}^{(d-\delta)} \times \mathcal{O}^{(\delta)}. \quad (2.16)$$

If satisfied, this condition amounts to the energy-momentum tensor being improvable, so that it is not only conserved and symmetric, but also traceless. As such, Polchinski's analysis [7] applies and so we set

$$\mathcal{W}_{\lambda\alpha\alpha} = \partial_\tau \mathcal{H}_{\tau\lambda} \quad (2.17)$$

and then solve (2.13) and (2.17) by taking<sup>1</sup>:

$$\begin{aligned} \partial_\lambda \mathcal{W}_{\lambda\alpha\beta} &= \frac{1}{2-d}(\partial_\alpha \partial_\tau \mathcal{H}_{\tau\beta} + \partial_\beta \partial_\tau \mathcal{H}_{\tau\alpha} - \partial^2 \mathcal{H}_{\alpha\beta} - \delta_{\alpha\beta} \partial_\tau \partial_\lambda \mathcal{H}_{\tau\lambda}) \\ &\quad + \frac{1}{(2-d)(d-1)}(\delta_{\alpha\beta} \partial^2 - \partial_\alpha \partial_\beta) \mathcal{H}_{\tau\tau} \end{aligned} \quad \text{for } d > 2, \quad (2.18a)$$

$$\partial_\lambda \mathcal{W}_{\lambda\alpha\beta} = \frac{1}{1-d}(\partial_\alpha \partial_\beta - \delta_{\alpha\beta} \partial^2) \mathcal{H} \quad \text{for } d = 2, \quad (2.18b)$$

where, in  $d = 2$ ,  $\mathcal{H}_{\tau\lambda} = \delta_{\tau\lambda} \mathcal{H}$ . Note that any terms containing two or more total derivatives in (2.15) can be absorbed into the final term, via a choice of  $\mathcal{H}_{\tau\lambda}$ ; this will be exploited in subsequent sections.

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<sup>1</sup> For non-unitary theories in  $d > 3$ , it is conceivable that there is an ambiguity in the energy-momentum tensor. This arises since a conformal primary may exist from which a conserved, symmetric, identically traceless tensor can be constructed [4, 8]. Such an ambiguity will not be explicitly treated in this paper.



Concretely,  $\mathcal{H}_{\tau\lambda}$  is obtained by substituting (2.17) into (2.16):

$$-\partial_\lambda \partial_\tau \mathcal{H}_{\tau\lambda} = \overline{\mathcal{T}}_{\alpha\alpha} + \partial_\lambda (\mathcal{F}_{\alpha\lambda\alpha} - \mathcal{F}_{\alpha\alpha\lambda}) + \delta \hat{\mathcal{O}}^{(d-\delta)} \times \mathcal{O}^{(\delta)}. \quad (2.19)$$

For this equation to be soluble, it seems there must be an interesting conspiracy, since the right-hand side is not manifestly  $\mathcal{O}(\partial^2)$ . The resolution is as follows: integrating this equation yields

$$\int \overline{\mathcal{T}}_{\alpha\alpha} + \delta \hat{\mathcal{O}}^{(d-\delta)} \cdot \mathcal{O}^{(\delta)} = 0, \quad (2.20)$$

whereas first multiplying by  $x_\mu$  yields

$$X_\mu \cdot \overline{\mathcal{T}}_{\alpha\alpha} - \int (\mathcal{F}_{\alpha\mu\alpha} - \mathcal{F}_{\alpha\alpha\mu}) + \delta \hat{\mathcal{O}}^{(d-\delta)} X_\mu \cdot \mathcal{O}^{(\delta)} = 0. \quad (2.21)$$

As we will explicitly confirm later within both classical and ERG analyses, (2.20) enforces dilatation invariance of the action whereas (2.21) enforces special conformal invariance. This is to be expected: improvement of the energy-momentum tensor implies full conformal symmetry.

Actually, there is an interesting subtlety in  $d = 2$ : it is possible to have a theory for which the action satisfies both dilatation and special conformal invariance but, nevertheless, the theory is not conformal! This can arise if it is not possible to express  $\mathcal{H}_{\tau\lambda} = \delta_{\tau\lambda} \mathcal{H}$ ; we shall encounter this later when examining the higher derivative  $\varphi \partial^4 \varphi$  theory, which suffers from particularly bad infrared behaviour in  $d = 2$ .

## B. Classical Theories

### 1. Analysis

In this section we apply the general methodology of section II A in a classical context. On the one hand, this will provide some experience with the advocated approach; on the other the results of this section will form part of the full quantum field theoretic result. Classically, the appropriate form for the defining equations of the energy-momentum tensor, (2.1a), (2.1b) and (2.1c) are:

$$\partial_\alpha t_{\alpha\beta} = -\partial_\beta \varphi \times \frac{\delta S}{\delta \varphi}, \quad (2.22a)$$

$$t_{\alpha\beta} = t_{\beta\alpha}, \quad (2.22b)$$

$$t_{\alpha\alpha} = -\delta \varphi \times \frac{\delta S}{\delta \varphi}, \quad (2.22c)$$

where we use  $t$  denotes the classical energy-momentum tensor. Note that these equations are exactly of the form as in the introduction, (1.11a), (1.11b) and (1.11c). Throughout this section, when we talk of an energy-momentum tensor which is conserved/traceless, we mean that it is conserved/traceless up to terms which vanish on the equations of motion.

To proceed, let us take  $\hat{L}$  to be an arbitrary element of the equivalence class of objects that integrate to the Wilsonian effective action, viz.

$$\int d^d x \hat{L}(x) = S[\varphi]. \quad (2.23)$$

For example, the Gaussian theory has  $\{\hat{L}\} = \{\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi, -\frac{1}{2}\varphi\partial^2\varphi\}$ , with the former being picked out as *the* Lagrangian. We henceforth demand quasi-locality, meaning that we restrict our attention to Lagrangians (and hence actions) which exhibit a derivative expansion, viz.

$$\hat{L}(x) = V(\varphi) + Z(\varphi)\partial_\mu\varphi\partial_\mu\varphi + \dots, \quad (2.24)$$

where  $V$  and  $Z$  do not contain any derivatives, and the ellipsis denotes terms higher order in derivatives.

To aid the analysis, define:

$$D_{\underline{\alpha}_j^i} \equiv \begin{cases} \prod_{k=j}^i \partial_{\alpha_k} & i \geq j, \\ 1 & i < j. \end{cases} \quad (2.25)$$

Using this notation, we have:

$$\frac{\delta S}{\delta\varphi} = \sum_{i=0}^{\infty} (-1)^i D_{\underline{\alpha}_1^i} \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)}. \quad (2.26)$$

Defining

$$S_\alpha = - \sum_{i=0}^{\infty} (-1)^i D_{\underline{\alpha}_1^i} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\alpha}_1^i} \varphi)}, \quad (2.27)$$

it is apparent that

$$\partial_\alpha S_\alpha = \frac{\delta S}{\delta\varphi} - \frac{\partial \hat{L}}{\partial\varphi}. \quad (2.28)$$

Courtesy of the chain rule,

$$\partial_\alpha \hat{L} = \sum_{i=0}^{\infty} \partial_\alpha D_{\underline{\alpha}_1^i} \varphi \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)}. \quad (2.29)$$

Turning out attention to the conservation equation (2.22a) the above results imply that

$$\begin{aligned} \partial_\beta \varphi \times \frac{\delta S}{\delta \varphi} &= \partial_\alpha (\partial_\beta \varphi \times S_\alpha + \delta_{\alpha\beta} \hat{L}) \\ &+ \partial_\alpha \partial_\beta \varphi \sum_{i=0}^{\infty} (-1)^i D_{\underline{\sigma}_1^i} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)} - \sum_{i=1}^{\infty} \partial_\beta D_{\underline{\sigma}_1^i} \varphi \frac{\partial \hat{L}}{\partial (D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.30)$$

In order to generate a contribution to the energy-momentum tensor, we must convert the whole of the right-hand side into a total derivative. As preparation for manipulating the middle term, observe that for some  $A$ ,  $B$  and  $j < i$

$$D_{\underline{\sigma}_1^j} A \times D_{\underline{\sigma}_1^{i+1}} B = \partial_{\sigma_{j+1}} (D_{\underline{\sigma}_1^j} A \times D_{\underline{\sigma}_1^{i+2}} B) - D_{\underline{\sigma}_1^{j+1}} A \times D_{\underline{\sigma}_1^{i+2}} B. \quad (2.31)$$

Feeding this result back into the final term and iterating yields:

$$D_{\underline{\sigma}_1^j} A \times D_{\underline{\sigma}_1^{i+1}} B = (-1)^{i+j} D_{\underline{\sigma}_1^i} A \times B - \sum_{k=j+1}^i (-1)^{j+k} \partial_{\sigma_k} (D_{\underline{\sigma}_1^{k-1}} A \times D_{\underline{\sigma}_1^{i+1}} B). \quad (2.32)$$

Applying this result—with  $j = 0$ —to the middle term of (2.30) gives:

$$\begin{aligned} \partial_\alpha \partial_\beta \varphi \sum_{i=0}^{\infty} (-1)^i D_{\underline{\sigma}_1^i} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)} &= \sum_{i=0}^{\infty} \partial_\beta \partial_\alpha D_{\underline{\sigma}_1^i} \varphi \times \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)} \\ &- \sum_{i=1}^{\infty} \sum_{j=1}^i (-1)^{i+j} \partial_{\sigma_j} \left( D_{\underline{\sigma}_1^{j-1}} \partial_\alpha \partial_\beta \varphi \times D_{\underline{\sigma}_1^{i+1}} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)} \right). \end{aligned} \quad (2.33)$$

The first term cancels the final one of (2.30) leaving, after some relabelling of indices:

$$\partial_\beta \varphi \times \frac{\delta S}{\delta \varphi} = \partial_\alpha \left( \partial_\beta \varphi \times S_\alpha + \delta_{\alpha\beta} \hat{L} - \sum_{i=1}^{\infty} \sum_{j=1}^i (-1)^{i+j} D_{\underline{\sigma}_1^j} \partial_\beta \varphi \times D_{\underline{\sigma}_1^{i+1}} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)} \right). \quad (2.34)$$

Using (2.27) to substitute for  $S_\alpha$ , the resulting term can be absorbed into the final one above by replacing both lower limits of the sums over  $j$  and  $i$  with zero. Recalling (2.4), we deduce the following contribution to the energy-momentum tensor:

$$\bar{t}_{\alpha\beta} = -\delta_{\alpha\beta} \hat{L} + \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^{i+j} D_{\underline{\sigma}_1^j} \partial_\beta \varphi \times D_{\underline{\sigma}_1^{i+1}} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)}. \quad (2.35)$$

To extract  $f_{\lambda\alpha\beta}$  (i.e. the classical version of  $\mathcal{F}_{\lambda\alpha\beta}$ ) we need to split  $\bar{t}_{\alpha\beta}$  into a symmetric piece plus a total derivative, as in (2.6). Utilizing (2.32) gives:

$$\begin{aligned} \bar{t}_{\alpha\beta} &= -\delta_{\alpha\beta} \hat{L} + \sum_{i=0}^{\infty} (i+1) D_{\underline{\sigma}_1^i} \partial_\beta \varphi \times \frac{\partial \hat{L}}{\partial (D_{\underline{\sigma}_1^i} \partial_\alpha \varphi)} \\ &- \partial_\lambda \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \sum_{k=j+1}^i (-1)^{i+k} \delta_{\lambda\sigma_k} D_{\underline{\sigma}_1^{k-1}} \partial_\beta \varphi \times D_{\underline{\sigma}_1^{i+1}} \frac{\partial \hat{L}}{\partial (\partial_\alpha D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.36)$$

The key point is that the first two terms are both symmetric under  $\alpha \leftrightarrow \beta$ . This is manifest for  $\delta_{\alpha\beta} \hat{L}$ . For the second term, we argue as follows. First, observe that

$$\sum_{i=0}^{\infty} (i+1) D_{\underline{\alpha}_1^i} \partial_{\beta} \varphi \times \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \partial_{\alpha} \varphi)} = \sum_{i=1}^{\infty} [D_{\underline{\alpha}_1^i}, x_{\alpha} \partial_{\beta}] \varphi \times \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)}. \quad (2.37)$$

With this in mind, we utilize rotational invariance of the action:

$$\begin{aligned} & \int d^d x (x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}) \varphi \frac{\delta S}{\delta \varphi} = 0 \\ \Rightarrow & \int d^d x \sum_{i=0}^{\infty} (x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}) D_{\underline{\alpha}_1^i} \varphi \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)} + \int d^d x \sum_{i=1}^{\infty} [D_{\underline{\alpha}_1^i}, x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}] \varphi \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)} = 0. \end{aligned} \quad (2.38)$$

The first term vanishes after using the chain rule and integrating by parts and so we conclude that rotational invariance alone is sufficient to ensure that the integrand of the final piece vanishes, at least up to total derivative terms. However, we can go further by exploiting quasi-locality of  $\hat{L}$ . Recall that this is a statement that  $\hat{L}$  has a derivative expansion; of course, since  $\hat{L}$  is a scalar, all partial derivatives must be paired up. In the integrand under analysis, the effect on  $\hat{L}$  of the  $x_{\alpha} \partial_{\beta}$  term is to split all such pairs up into  $\partial_{\alpha}$  and  $\partial_{\beta}$ . Since all possible splittings are summed over, the result is symmetric under interchange of indices and so we conclude that

$$\sum_{i=1}^{\infty} [D_{\underline{\alpha}_1^i}, x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}] \varphi \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)} = 0. \quad (2.39)$$

Returning to (2.36), we use the recipe (2.6) to extract

$$\bar{t}_{\alpha\beta}^{\text{sym}} = -\delta_{\alpha\beta} \hat{L} + \sum_{i=1}^{\infty} [D_{\underline{\alpha}_1^i}, x_{\alpha} \partial_{\beta}] \varphi \times \frac{\partial \hat{L}}{\partial (D_{\underline{\alpha}_1^i} \varphi)}, \quad (2.40a)$$

$$f_{\lambda\alpha\beta} = - \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \sum_{k=j+1}^i (-1)^{i+k} \delta_{\lambda\sigma_k} D_{\underline{\alpha}_1^{k-1}} \partial_{\beta} \varphi \times D_{\underline{\alpha}_{k+1}^i} \frac{\partial \hat{L}}{\partial (\partial_{\alpha} D_{\underline{\alpha}_1^i} \varphi)}, \quad (2.40b)$$

from which we can construct the full energy-momentum tensor according to (2.14), at least up to the conserved, symmetric  $\partial_{\lambda} w_{\lambda\alpha\beta}$ . To determine the latter, we take the trace—for which we can use (2.15)—and compare with (2.22c). Mimicking section II A, we take

$$\partial_\lambda w_{\lambda\alpha\alpha} = \partial_\lambda \partial_\tau h_{\tau\lambda}:$$

$$\begin{aligned} t_{\alpha\alpha} &= \partial_\lambda \partial_\tau h_{\tau\lambda} - d\hat{L} + \sum_{i=1}^{\infty} [D_{\underline{\sigma}_1^i}, x \cdot \partial] \varphi \times \frac{\partial \hat{L}}{\partial(D_{\underline{\sigma}_1^i} \varphi)} \\ &\quad + (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \partial_\lambda \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \sum_{k=j+1}^i (-1)^{i+k} D_{\underline{\sigma}_1^{k-1}} \partial_\omega \varphi \times D_{\underline{\sigma}_1^i} \frac{\partial \hat{L}}{\partial(\partial_\rho \partial_\sigma D_{\underline{\sigma}_1^{k-1}} D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.41)$$

Rather than jumping straight to (2.19), we recall the comments under (2.18a) and (2.18b): the game now is to simplify this expression by absorbing various  $O(\partial^2)$  terms into the first term on the right-hand side. We begin by re-expressing:

$$\begin{aligned} (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \partial_\lambda \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \sum_{k=j+1}^i D_{\underline{\sigma}_1^{k-1}} D_{\underline{\sigma}_1^i} \partial_\omega \varphi \times \frac{\partial \hat{L}}{\partial(\partial_\rho \partial_\sigma D_{\underline{\sigma}_1^{k-1}} D_{\underline{\sigma}_1^i} \varphi)} \\ = \frac{1}{2} (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \partial_\lambda \sum_{i=2}^{\infty} [[D_{\underline{\sigma}_1^i}, x_\sigma], x_\rho \partial_\omega] \varphi \times \frac{\partial \hat{L}}{\partial(D_{\underline{\sigma}_1^i} \varphi)}, \end{aligned} \quad (2.42)$$

which can be used to obtain

$$\begin{aligned} t_{\alpha\alpha} &= \partial_\lambda \partial_\tau \tilde{h}_{\tau\lambda} - d\hat{L} + \sum_{i=1}^{\infty} [D_{\underline{\sigma}_1^i}, x \cdot \partial] \varphi \times \frac{\partial \hat{L}}{\partial(D_{\underline{\sigma}_1^i} \varphi)} \\ &\quad + \frac{1}{2} (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \partial_\lambda \sum_{i=2}^{\infty} [[D_{\underline{\sigma}_1^i}, x_\sigma], x_\rho \partial_\omega] \varphi \times \frac{\partial \hat{L}}{\partial(D_{\underline{\sigma}_1^i} \varphi)}, \end{aligned} \quad (2.43)$$

where

$$\begin{aligned} h_{\tau\lambda} &= \tilde{h}_{\tau\lambda} + (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \sum_{i=2}^{\infty} \sum_{j=0}^{i-2} \sum_{k=j+1}^{i-1} \sum_{l=k+1}^i (-1)^{i+l} D_{\underline{\sigma}_1^{k-1}} D_{\underline{\sigma}_1^{l-1}} \partial_\omega \varphi \\ &\quad \times D_{\underline{\sigma}_1^i} \frac{\partial \hat{L}}{\partial(\partial_\rho \partial_\sigma \partial_\tau D_{\underline{\sigma}_1^{k-1}} D_{\underline{\sigma}_1^{l-1}} D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.44)$$

Comparing (2.43) with (2.22c) we arrive at the following constraint for CFTs:

$$\begin{aligned} -\partial_\lambda \partial_\tau \tilde{h}_{\tau\lambda} &= -d\hat{L} + \sum_{i=1}^{\infty} [D_{\underline{\sigma}_1^i}, x \cdot \partial] \varphi \times \frac{\partial \hat{L}}{\partial(D_{\underline{\sigma}_1^i} \varphi)} + \delta\varphi \times \frac{\delta S}{\delta\varphi} \\ &\quad + \frac{1}{2} (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \partial_\lambda \sum_{i=2}^{\infty} [[D_{\underline{\sigma}_1^i}, x_\sigma], x_\rho \partial_\omega] \varphi \times \frac{\partial \hat{L}}{\partial(D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.45)$$

As mentioned at the end of section II A, there is a neat way to check the consistency of this equation. First of all, we integrate it directly. The total derivative terms vanish. To process

the surviving terms, observe that

$$\begin{aligned} \int d^d x \, x \cdot \partial \varphi \frac{\delta S}{\delta \varphi} &= \int d^d x \, x \cdot \partial \varphi \sum_{i=0}^{\infty} (-1)^i D_{\underline{\sigma}_1^i} \frac{\partial \hat{L}}{\partial (D_{\underline{\sigma}_1^i} \varphi)} \\ &= \int d^d x \left( \sum_{i=0}^{\infty} x \cdot \partial D_{\underline{\sigma}_1^i} \varphi + \sum_{i=1}^{\infty} [D_{\underline{\sigma}_1^i}, x \cdot \partial] \varphi \right) \frac{\partial \hat{L}}{\partial (D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.46)$$

The first term can be processed by the chain rule to give  $-dS$ . It is thus apparent that, upon integration, (2.45) reduces to

$$D^{(\delta)} \varphi \cdot \frac{\delta S}{\delta \varphi} = 0, \quad (2.47)$$

which is of course nothing but the statement of dilatation invariance.

Returning to (2.45), we now multiply by  $2x_\mu$  and then integrate. To see what this gives, observe that

$$\begin{aligned} K^{(0)}_\mu \varphi \cdot \frac{\delta S}{\delta \varphi} &= \int d^d x \left( \sum_{i=0}^{\infty} (2x_\mu x \cdot \partial - x^2 \partial_\mu) D_{\underline{\sigma}_1^i} \varphi + \sum_{i=1}^{\infty} [D_{\underline{\sigma}_1^i}, 2x_\mu x \cdot \partial - x^2 \partial_\mu] \varphi \right) \frac{\partial \hat{L}}{\partial (D_{\underline{\sigma}_1^i} \varphi)} \\ &= -2d \int d^d x \, x_\mu \hat{L} - (\delta_{\omega\mu} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\mu}) \int d^d x \sum_{i=1}^{\infty} [D_{\underline{\sigma}_1^i}, x_\sigma x_\rho \partial_\omega] \varphi \frac{\partial \hat{L}}{\partial (D_{\underline{\sigma}_1^i} \varphi)}. \end{aligned} \quad (2.48)$$

The final term can be processed by manipulating the commutator:

$$\begin{aligned} [D_{\underline{\sigma}_1^i}, x_\sigma x_\rho \partial_\omega] &= [D_{\underline{\sigma}_1^i}, x_\sigma] x_\rho \partial_\omega + x_\sigma [D_{\underline{\sigma}_1^i}, x_\rho] \partial_\omega \\ &= [[D_{\underline{\sigma}_1^i}, x_\sigma], x_\rho] \partial_\omega + x_\rho [D_{\underline{\sigma}_1^i}, x_\sigma] \partial_\omega + x_\sigma [D_{\underline{\sigma}_1^i}, x_\rho] \partial_\omega. \end{aligned} \quad (2.49)$$

From this it follows that

$$\begin{aligned} (\delta_{\omega\mu} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\mu}) [D_{\underline{\sigma}_1^i}, x_\sigma x_\rho \partial_\omega] &= 2x_\rho [D_{\underline{\sigma}_1^i}, x_\rho \partial_\mu - x_\mu \partial_\rho] + \\ &\quad (\delta_{\omega\mu} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\mu}) [[D_{\underline{\sigma}_1^i}, x_\sigma], x_\rho] \partial_\omega - 2x_\mu [D_{\underline{\sigma}_1^i}, x \cdot \partial]. \end{aligned} \quad (2.50)$$

When substituted into (2.48), the first term on the right-hand side vanishes as a consequence of (2.39); returning to (2.45), we thus conclude that multiplying by  $x_\mu$  and integrating implies:

$$K^{(\delta)}_\mu \varphi \cdot \frac{\delta S}{\delta \varphi} = 0 \quad (2.51)$$

which, in combination with (2.47), shows that (2.45) encodes invariance under both dilatations and special conformal transformations.

## 2. Examples

*a. Gaussian Theory* To get a feeling for the construction of the energy-momentum tensor including, in particular (2.45), consider the case of the Gaussian fixed-point. To start with, take  $\hat{L} = \frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi$ . Referring back to (2.40a) and (2.40b) it is apparent that

$$\bar{t}_{\alpha\beta}^{\text{sym}} = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}\delta_{\alpha\beta}\partial_\mu\varphi\partial_\mu\varphi, \quad f_{\lambda\alpha\beta} = 0. \quad (2.52)$$

From (2.44) we see that  $h_{\tau\lambda} = \tilde{h}_{\tau\lambda}$ , with the latter determined by (2.45)

$$\begin{aligned} -\partial_\lambda\partial_\tau\tilde{h}_{\tau\lambda} &= -\delta_0\partial_\lambda\varphi\partial_\lambda\varphi - \delta\varphi\partial^2\varphi \\ &= -\frac{\delta}{2}\partial^2\varphi^2 + (\delta - \delta_0)\partial_\lambda\varphi\partial_\lambda\varphi, \end{aligned} \quad (2.53)$$

therefore implying that  $\delta = \delta_0$ , as expected, with

$$h_{\tau\lambda} = \tilde{h}_{\tau\lambda} = \frac{d-2}{4}\delta_{\tau\lambda}\varphi^2. \quad (2.54)$$

To construct the full energy-momentum tensor we employ (2.18a) and (2.18b); conveniently, for the present case where  $h_{\tau\lambda} \sim \delta_{\tau\lambda}$ , the former reduces to the latter and so we find, as expected:

$$t_{\alpha\beta} = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}\delta_{\alpha\beta}\partial_\mu\varphi\partial_\mu\varphi + \frac{d-2}{4(d-1)}(\delta_{\alpha\beta}\partial^2 - \partial_\alpha\partial_\beta)\varphi^2. \quad (2.55)$$

It is easy to check that taking  $\hat{L} = -\frac{1}{2}\varphi\partial^2\varphi$  gives the same result.

*b. Higher Derivative Theory* As a slightly more involved example, we will explore the free theory with a kinetic term quartic in derivatives. To begin with, we shall consider  $\hat{L} = \frac{1}{2}\partial_\mu\partial_\nu\varphi\partial_\mu\partial_\nu\varphi$ . Power counting informs us that the scaling dimension of  $\varphi$  is  $(d-4)/2$  and so the theory is non-unitary. Nevertheless, at least for  $d > 2$ , we can construct the energy-momentum tensor. Referring back to (2.40a) and (2.40b), we have:

$$\bar{t}_{\alpha\beta}^{\text{sym}} = -\frac{1}{2}\delta_{\alpha\beta}\partial_\mu\partial_\nu\varphi\partial_\mu\partial_\nu\varphi + 2\partial_\alpha\partial_\nu\varphi\partial_\beta\partial_\nu\varphi, \quad (2.56a)$$

$$f_{\lambda\alpha\beta} = -\partial_\beta\varphi\partial_\alpha\partial_\lambda\varphi, \quad (2.56b)$$

corresponding to the conserved but not symmetric tensor

$$\bar{t}_{\alpha\beta} = \partial_\mu\partial_\alpha\varphi\partial_\mu\partial_\beta\varphi - \frac{1}{2}\delta_{\alpha\beta}\partial_\mu\partial_\nu\varphi\partial_\mu\partial_\nu\varphi - \partial_\beta\varphi\partial_\alpha\partial^2\varphi. \quad (2.57)$$

Following the recipe (2.14) yields the conserved, symmetric tensor

$$t_{\alpha\beta} = \partial_\mu\varphi\partial_\mu\partial_\alpha\partial_\beta\varphi + \partial^2\varphi\partial_\alpha\partial_\beta\varphi - \frac{1}{2}\delta_{\alpha\beta}\partial_\mu\partial_\nu\varphi\partial_\mu\partial_\nu\varphi - \partial_\beta\varphi\partial_\alpha\partial^2\varphi - \partial_\alpha\varphi\partial_\beta\partial^2\varphi + \partial_\lambda w_{\lambda\alpha\beta}, \quad (2.58)$$

where  $\partial_\lambda w_{\lambda\alpha\beta}$  is determined by (2.18a) in terms of  $h_{\tau\lambda}$ . From (2.44), in this particular case  $h_{\tau\lambda} = \tilde{h}_{\tau\lambda}$  and from (2.45)

$$-\partial_\lambda \partial_\tau \tilde{h}_{\tau\lambda} = \frac{4-d}{2} \partial_\mu \partial_\nu \varphi \partial_\mu \partial_\nu \varphi + \delta\varphi \partial^4 \varphi + \partial_\mu (\partial_\mu \varphi \partial^2 \varphi - 2\partial_\nu \varphi \partial_\mu \partial_\nu \varphi). \quad (2.59)$$

Components of the first two terms can be transferred to the final term by writing

$$\partial_\mu \partial_\nu \varphi \partial_\mu \partial_\nu \varphi = \partial_\mu (\partial_\nu \varphi \partial_\mu \partial_\nu \varphi) - \partial_\mu \varphi \partial_\mu \partial^2 \varphi, \quad (2.60a)$$

$$\varphi \partial^4 \varphi = \partial_\mu (\varphi \partial_\mu \partial^2 \varphi) - \partial_\mu \varphi \partial_\mu \partial^2 \varphi \quad (2.60b)$$

so that we have

$$-\partial_\lambda \partial_\tau \tilde{h}_{\tau\lambda} = -\frac{2+\eta}{2} \partial_\mu \varphi \partial_\mu \partial^2 \varphi + \partial_\mu (\partial_\mu \varphi \partial^2 \varphi - d/2 \partial_\nu \varphi \partial_\mu \partial_\nu \varphi + \delta\varphi \partial_\mu \partial^2 \varphi). \quad (2.61)$$

The final term can be massaged to give

$$\partial_\mu \partial_\nu (\partial_\mu \varphi \partial_\nu \varphi + \delta\varphi \partial_\mu \partial_\nu \varphi) - \frac{2d+\eta}{4} \partial^2 (\partial_\mu \varphi \partial_\mu \varphi),$$

and so we conclude that  $\eta = -2$ , as expected, and, for  $d > 2$ ,

$$h_{\tau\lambda} = \tilde{h}_{\tau\lambda} = \frac{d-1}{2} \delta_{\tau\lambda} \partial_\mu \varphi \partial_\mu \varphi - \partial_\lambda \varphi \partial_\tau \varphi - \frac{d-4}{2} \varphi \partial_\lambda \partial_\tau \varphi. \quad (2.62)$$

By inspection of (2.18b) it is apparent that the solution for  $h_{\tau\lambda}$  does not exist in  $d = 2$ . It is straightforward to check that

$$t_{\alpha\alpha} = -\frac{d-4}{2} \varphi \partial^4 \varphi, \quad (2.63)$$

as expected. Constructing  $\partial_\lambda w_{\lambda\alpha\beta}$  from (2.18a), substituting into (2.66) and simplifying yields the full (if rather unwieldy) energy-momentum tensor:

$$\begin{aligned} t_{\alpha\beta} = \frac{1}{2(d-2)(d-1)} \Big\{ & (4d-8) \partial_\mu \varphi \partial_\mu \partial_\alpha \partial_\beta \varphi + d(d+2) \partial^2 \varphi \partial_\alpha \partial_\beta \varphi \\ & + (d-2)(d-4) \varphi \partial_\alpha \partial_\beta \partial^2 \varphi + (4-d^2) (\partial_\alpha \varphi \partial_\beta \partial^2 \varphi + \partial_\beta \varphi \partial_\alpha \partial^2 \varphi) - 4d \partial_\alpha \partial_\mu \varphi \partial_\beta \partial_\mu \varphi \\ & + \delta_{\alpha\beta} [2\partial_\mu \partial_\nu \varphi \partial_\mu \partial_\nu \varphi + (2d-4) \partial_\mu \varphi \partial_\mu \partial^2 \varphi - (d+2) \partial^2 \varphi \partial^2 \varphi - (d-2)(d-4) \varphi \partial^4 \varphi] \Big\}. \end{aligned} \quad (2.64)$$

The expression is manifestly symmetric and it is easy to check that it is conserved and traceless.



A variant of the above analysis which exercises all terms involved in the construction of the energy-momentum tensor is achieved by taking instead  $\hat{L} = -\frac{1}{2}\partial_\mu\varphi\partial_\mu\partial^2\varphi$ . In this case we find:

$$\bar{t}_{\alpha\beta}^{\text{sym}} = \frac{1}{2}\delta_{\alpha\beta}\partial_\mu\varphi\partial_\mu\partial^2\varphi - \frac{1}{2}\partial_\alpha\varphi\partial_\beta\partial^2\varphi - \frac{1}{2}\partial_\beta\varphi\partial_\alpha\partial^2\varphi - \partial_\mu\varphi\partial_\mu\partial_\alpha\partial_\beta\varphi, \quad (2.65a)$$

$$f_{\lambda\alpha\beta} = -\frac{2}{3}\partial_\beta\varphi\partial_\lambda\partial_\alpha\varphi - \frac{1}{3}\delta_{\alpha\lambda}\partial_\beta\varphi\partial^2\varphi + \frac{2}{3}\partial_\lambda\partial_\beta\varphi\partial_\alpha\varphi + \frac{2}{3}\partial_\alpha\partial_\beta\varphi\partial_\lambda\varphi + \frac{2}{3}\delta_{\alpha\lambda}\partial_\mu\partial_\beta\varphi\partial_\mu\varphi; \quad (2.65b)$$

it can be checked that the sum of these terms,  $\bar{t}_{\alpha\beta}$  is conserved. According to the recipe (2.14), we construct the conserved, symmetric tensor

$$t_{\alpha\beta} = \frac{1}{6}\delta_{\alpha\beta}\partial^2\varphi\partial^2\varphi + \frac{1}{3}\delta_{\alpha\beta}\partial_\mu\varphi\partial_\mu\partial^2\varphi + \frac{2}{3}\partial_\alpha\partial_\beta\varphi\partial^2\varphi + \frac{2}{3}\partial_\mu\varphi\partial_\mu\partial_\alpha\partial_\beta\varphi \\ - \frac{1}{3}\delta_{\alpha\beta}\partial_\mu\partial_\nu\varphi\partial_\mu\partial_\nu\varphi - \partial_\alpha\varphi\partial_\beta\partial^2\varphi - \partial_\beta\varphi\partial_\alpha\partial^2\varphi + \partial_\lambda w_{\lambda\alpha\beta}. \quad (2.66)$$

Contrary to the previous analysis,  $h_{\tau\lambda}$  is non-trivially related to  $\tilde{h}_{\tau\lambda}$ :

$$h_{\tau\lambda} = \tilde{h}_{\tau\lambda} - \frac{d-2}{6}\partial_\tau\varphi\partial_\lambda\varphi + \frac{1}{3}\delta_{\tau\lambda}\partial_\mu\varphi\partial_\mu\varphi; \quad (2.67)$$

and from (2.45)

$$-\partial_\lambda\partial_\tau\tilde{h}_{\tau\lambda} = \frac{d-4}{2}\partial_\mu\varphi\partial_\mu\partial^2\varphi + \delta\varphi\partial^4\varphi + \partial_\mu(\partial_\mu\varphi\partial^2\varphi - \delta_0\partial_\nu\varphi\partial_\mu\partial_\nu\varphi). \quad (2.68)$$

Utilizing (2.60b),

$$-\partial_\lambda\partial_\tau\tilde{h}_{\tau\lambda} = -\frac{2+\eta}{2}\partial_\mu\varphi\partial_\mu\partial^2\varphi + \partial_\mu(\partial_\mu\varphi\partial^2\varphi - \delta_0\partial_\nu\varphi\partial_\mu\partial_\nu\varphi + \delta\varphi\partial_\mu\partial^2\varphi) \quad (2.69)$$

and so, as before,  $\eta = -2$ . After a bit of rearrangement (and utilizing the result for  $\eta$ ) we find, for  $d > 2$ :

$$\tilde{h}_{\tau\lambda} = \frac{d-2}{2}\delta_{\tau\lambda}\partial_\nu\varphi\partial_\nu\varphi - \partial_\lambda\varphi\partial_\tau\varphi - \frac{d-4}{2}\varphi\partial_\lambda\partial_\tau\varphi, \quad (2.70)$$

which yields

$$h_{\tau\lambda} = \frac{3d-4}{6}\delta_{\tau\lambda}\partial_\mu\varphi\partial_\mu\varphi - \frac{d+4}{6}\partial_\tau\varphi\partial_\lambda\varphi - \frac{d-4}{2}\varphi\partial_\tau\partial_\lambda\varphi. \quad (2.71)$$

As before, it is easy to confirm (2.63) but somewhat involved to reconstruct the full energy-momentum tensor (2.64).

*c. Interacting Theories* A crucial feature of the equations which define the classical energy momentum tensor, (2.22a), (2.22b) and (2.22c), is that contributions to the action depending on powers of the field decouple from one another. However, consistency between these various terms is enforced by the final condition encoding dilatation invariance. For example, consider adding a potential term to the gaussian theory:

$$\hat{L} = \partial_\mu \varphi \partial_\mu \varphi + V(\varphi). \quad (2.72)$$

The energy-momentum tensor for this theory is given by:

$$t_{\alpha\beta} = t_{\alpha\beta}^{\text{Gauss}} + \delta_{\alpha\beta} V(\varphi), \quad (2.73)$$

with dilatation invariance requiring that, for  $d > 2$ ,

$$V(\varphi) \propto \varphi^{d/\delta_0}. \quad (2.74)$$

This is very different from the quantum case, where the right-hand side of the analogues of (2.22a) and (2.22c) are quadratic both in the action and also functional derivatives, making the problem of finding explicit solutions very much more difficult.

## C. ERG Representation

### 1. Notation and Conventions

To formulate the ERG equation, we introduce an ultraviolet cutoff function which, as in the introduction, we denote by  $K((x-y)^2)$ . As with all ingredients of a good ERG equation this function must be quasi-local (cf. the discussion below (2.23)). From this we construct an object  $G$  satisfying

$$(d + x \cdot \partial_x + y \cdot \partial_y) K((x-y)^2) = \partial_x^2 G((x-y)^2). \quad (2.75)$$

This is perhaps more intuitive in momentum space<sup>2</sup> where it translates to  $p \cdot \partial_p K = p^2 G$  or just  $G(p^2) = 2 dK(p^2)/dp^2$ . From  $G$  it is useful to construct

$$G_\mu((x-y)^2) \equiv (x+y)_\mu G((x-y)^2). \quad (2.76)$$

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<sup>2</sup> We use the same symbol for a function of coordinates and its Fourier transform.

Taking Green's function to be  $\mathcal{G}_0$ , so that  $-\partial^2 \mathcal{G}_0 = \mathbb{1}$ , we define a ultraviolet regulated version

$$\mathcal{G} = \mathcal{G}_0 \cdot K \quad (2.77)$$

where, as alluded to in the introduction, we use the following shorthand for integrals:

$$\Psi \cdot \Phi \equiv \int d^d x \Psi(x) \Phi(x), \quad \Psi \cdot F \cdot \Phi \equiv \int d^d x d^d y \Psi(x) F(x, y) \Phi(y). \quad (2.78)$$

Now we have all the ingredients we need; the ERG equation and its partner encoding special conformal invariance read, up to vacuum terms (which are neglected throughout this paper):

$$\left\{ D^{(\delta)} \varphi \cdot \frac{\delta}{\delta \varphi} + \varphi \cdot \mathcal{G}^{-1} \cdot G \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \cdot \frac{\delta}{\delta \varphi} \right\} e^{-S} = 0, \quad (2.79a)$$

$$\left\{ K^{(\delta)}_{\mu} \varphi \cdot \frac{\delta}{\delta \varphi} + \varphi \cdot \mathcal{G}^{-1} \cdot G_{\mu} \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G_{\mu} \cdot \frac{\delta}{\delta \varphi} - \eta \partial_{\mu} \varphi \cdot K^{-1} \cdot G \cdot \frac{\delta}{\delta \varphi} \right\} e^{-S} = 0. \quad (2.79b)$$

The ERG equation is slightly different from that usually appearing the literature—which is the variant of the Wilson/Polchinski equations [9, 10] proposed in [11]—on account of it using the full Wilsonian effective action; the relationship to the more common form is given in appendix A. The special conformal equation (2.79b) is a similar re-expression of the equation written down in [4].

The analysis of the following section will utilize some new notation. To motivate this, let us anticipate that in the ERG treatment of the energy-momentum tensor we will encounter a term like

$$\partial_{\beta} \varphi \cdot K^{-1} \times K \cdot \frac{\delta S}{\delta \varphi} = \partial_{\beta} \varphi \times \frac{\delta S}{\delta \varphi} - \partial_{\beta} \varphi \cdot K^{-1} \cdot (K \times \mathbb{1} - \mathbb{1} \times K) \cdot \frac{\delta S}{\delta \varphi}. \quad (2.80)$$

The first term on the right-hand side we recognize from the classical analysis and so our task will be to process the second term. In particular, we would like to re-write it as a total derivative. To this end observe that, for some quasi-local  $F((x - y)^2)$ ,

$$F \times \mathbb{1} - \mathbb{1} \times F = \partial_{\alpha} \mathcal{F}_{\alpha}, \quad (2.81)$$

where  $\mathcal{F}_{\alpha}(y, z; x)$  is also quasi-local and the partial derivative on the right-hand side is understood to be with respect to  $x$ . This is easy to see by making making the coordinate dependence explicit and integrating:

$$\int d^d x (F(y, x) \delta^{(d)}(x - z) - \delta^{(d)}(y - x) F(x, z)) = 0$$

and, since  $F$  is quasi-local, (2.81) follows. Note that, courtesy of translation invariance,  $\mathcal{F}$  can be rewritten as a function of coordinate differences:  $\mathcal{F}_\alpha(y - x, z - x)$ . Here we are overloading notation so that the two argument form of  $\mathcal{F}$  is considered separate from the three argument form.

Were we to directly utilize (2.81) in (2.80), our notation would suffer from an ambiguity since the dot-notation suggests that all coordinates are integrated over. To retain a compact expression without having to pollute all of our equations with explicit coordinate dependence, we develop some new notation. Given test functions  $\Psi$  and  $\Phi$ , we hijack the symbols  $\bowtie$  and  $\ltimes$  as follows:

$$(\Psi \bowtie \mathcal{F} \ltimes \Phi)(x) \equiv \int d^d y d^d z \Psi(y) \mathcal{F}(y, z; x) \Phi(z). \quad (2.82)$$

The left-hand side has exactly the same meaning as  $\Psi \cdot \mathcal{F}(x) \cdot \Phi$ . However, whereas we must retain the  $x$  in the latter equation to avoid ambiguity, we will be able to use the former with the explicit coordinate dependence dropped.

The notation of (2.82) naturally extends to the case where  $\Phi$  and  $\Psi$  depend on two arguments, viz.

$$(\Psi \bowtie \mathcal{F} \ltimes \Phi)(u, v; x) \equiv \int d^d y d^d z \Psi(u, y) \mathcal{F}(y, z; x) \Phi(z, v); \quad (2.83)$$

clearly it can also be applied to the mixed case where  $\Phi$  has one argument but  $\Psi$  has two (or vice-versa).

To gain some experience with the new notation, let us record several useful properties:

$$\Psi \bowtie \partial_\alpha \mathcal{F} \ltimes \Phi = \partial_\alpha (\Psi \bowtie \mathcal{F} \ltimes \Phi) \quad (2.84a)$$

$$\Psi \bowtie \mathbb{1} \ltimes \mathcal{F} \ltimes \mathbb{1} \ltimes \Phi = \Psi \bowtie \mathcal{F} \ltimes \Phi, \quad (2.84b)$$

$$\mathbb{1} \bowtie \partial_\alpha \mathcal{F} \ltimes \mathbb{1} = -\partial_\alpha \mathbb{1} \ltimes \mathcal{F} \ltimes \mathbb{1} - \mathbb{1} \ltimes \mathcal{F} \ltimes \mathbb{1} \overleftarrow{\partial}_\alpha. \quad (2.84c)$$

The first equation is a straightforward consequence of (2.82). There are several ways to see the second equation. Most directly, one could simply substitute  $\delta$ -functions for the  $\mathbb{1}$ s and apply (2.83):

$$\mathbb{1} \bowtie \mathcal{F} \ltimes \mathbb{1} = \int d^d y d^d z \delta^{(d)}(u - y) \mathcal{F}(y, z; x) \delta^{(d)}(z - v) = \mathcal{F}(u, v; x); \quad (2.85)$$

after applying (2.82) again, (2.84b) follows. At a more heuristic level, one could swap the  $\bowtie$  and  $\ltimes$  for dots—mindful of the ambiguity above—and then convert them back again after eliding the  $\mathbb{1}$ s.

Equation (2.84c) exploits translation invariance of  $\mathcal{F}$ :

$$\begin{aligned}
& \int d^d y d^d z \delta^{(d)}(u - y) \frac{\partial}{\partial x_\mu} \mathcal{F}(y - x, z - x) \delta^{(d)}(v - z) \\
&= - \int d^d y d^d z \delta^{(d)}(u - y) \left( \frac{\partial}{\partial y_\mu} + \frac{\partial}{\partial z_\mu} \right) \mathcal{F}(y - x, z - x) \delta^{(d)}(z - v) \\
&= - \left( \frac{\partial}{\partial u_\mu} + \frac{\partial}{\partial v_\mu} \right) \int d^d y d^d z \delta^{(d)}(u - y) \mathcal{F}(y - x, z - x) \delta^{(d)}(z - v)
\end{aligned}$$

where, to go to the final line, we have integrated by parts and exploited translational invariance of the  $\delta$ -functions. The backward-pointing arrow in (2.84c) indicates that the associated operator—here a derivative—acts on the last argument of the object to its left, which in this case would correspond to  $\partial/\partial v_\alpha \delta^{(d)}(z, v)$ , with  $\delta^{(d)}(z, v) = \delta^{(d)}(z - v)$ .

The notation of (2.82) allows us to neatly express  $\mathcal{F}_\alpha$  in terms of derivatives of a scalar:

$$\mathcal{F}_\alpha = \partial_\alpha \mathbb{1} \times \mathcal{F} \times \mathbb{1} - \mathbb{1} \times \mathcal{F} \times \overleftarrow{\partial}_\alpha. \quad (2.86)$$

Integrating over test functions and employing (2.84b) gives the useful equations

$$\Psi \times \mathcal{F}_\alpha \times \Phi = -\partial_\alpha \Psi \times \mathcal{F} \times \Phi + \Psi \times \mathcal{F} \times \partial_\alpha \Phi \quad (2.87a)$$

$$= -\partial_\alpha (\Psi \times \mathcal{F} \times \Phi) + 2\Psi \times \mathcal{F} \times \partial_\alpha \Phi \quad (2.87b)$$

$$= \partial_\alpha (\Psi \times \mathcal{F} \times \Phi) - 2\partial_\alpha \Psi \times \mathcal{F} \times \Phi. \quad (2.87c)$$

To establish a relationship between  $\mathcal{F}$  and  $F$ , observe that:

$$\Psi \times \partial_\alpha \mathcal{F}_\alpha \times \Phi = \Psi \times \mathcal{F} \times \partial^2 \Phi - \partial^2 \Psi \times \mathcal{F} \times \Phi. \quad (2.88)$$

This can be compared with (2.81); it is particularly transparent to do so in momentum space, which yields:

$$\mathcal{F}(k - p, p) = \frac{F((k - p)^2) - F(p^2)}{k \cdot (k - 2p)}. \quad (2.89)$$

The expansion in  $k$  will play a key role later on; in its most useful form it is:

$$\mathcal{F}(k - p, p) = \frac{1}{2} [F'(p^2) + F'((k - p)^2)] + \mathcal{O}(k^2), \quad (2.90)$$

where the prime denotes a derivative with respect to the argument. Equivalently,

$$\begin{aligned}
\Psi \times \mathcal{F} \times \Phi &= \frac{1}{2} \Psi \cdot (F' \times \mathbb{1} + \mathbb{1} \times F') \cdot \Phi + \mathcal{O}(\partial^2) \\
&= \frac{1}{2} \Psi \cdot \{F', \mathbb{1}\} \cdot \Phi + \mathcal{O}(\partial^2)
\end{aligned} \quad (2.91)$$

where, in position space, we understand  $F'$  to be the Fourier transform of  $dF(p^2)/dp^2$ , and so forth. Equivalently, recalling (1.2), we can define  $F'$  via:

$$D^{(d/2)}F + F\overleftarrow{D}^{(d/2)} = 2\partial^2 F' \quad \text{or} \quad D^{(\delta_0)}\mathcal{G}_0 \cdot F + \mathcal{G}_0 \cdot F\overleftarrow{D}^{(\delta_0)} = -2F'. \quad (2.92)$$

## 2. Analysis

To specialize the general analysis of the energy-momentum tensor of section II A to the ERG requires expressions for the  $\mathcal{O}^{(\delta)}$  and  $\hat{\mathcal{O}}^{(d-\delta)}$  appearing in (2.1a) and (2.1c). First of all, we note the existence of a pair of primary fields [4, 12–14]

$$\mathcal{O}_{\text{ERG}}^{(\delta)} = K^{-1} \cdot \varphi - R \cdot \frac{\delta S}{\delta \varphi}, \quad (2.93a)$$

$$\mathcal{O}_{\text{ERG}}^{(d-\delta)} = \frac{\delta S}{\delta \varphi} \cdot K, \quad (2.93b)$$

where the subscript ERG is a reminder that we are in the (quasi-local) ERG representation and, in momentum space, for  $\eta < 2$

$$R(p^2) = p^{2(\eta/2-1)} K(p^2) \int_0^{p^2} dq^2 q^{-2(\eta/2)} \frac{d}{dq^2} \frac{1}{K(q^2)}. \quad (2.94)$$

The extension of  $\mathcal{O}_{\text{ERG}}^{(d-\delta)}$  to an  $\hat{\mathcal{O}}_{\text{ERG}}^{(d-\delta)}$  satisfying (2.3) is given simply by [4]

$$\hat{\mathcal{O}}_{\text{ERG}}^{(d-\delta)} = \frac{\delta S}{\delta \varphi} \cdot K - \frac{\delta}{\delta \varphi} \cdot K. \quad (2.95)$$

Note that, strictly speaking, if we wish to use the representation of  $\mathcal{D}$  and  $\mathcal{K}_\mu$  given by (1.5) and (1.7) then the  $\mathcal{O}^{(\delta)}$  and  $\mathcal{O}^{(d-\delta)}$  appearing in (2.2) and (2.3) are the ERG representations given above but multiplied by  $e^{-S}$ . Equivalently, we can stick with  $\mathcal{O}_{\text{ERG}}^{(\delta)}$  and  $\hat{\mathcal{O}}_{\text{ERG}}^{(d-\delta)}$  and take a representation of the dilatation operator give by  $\mathcal{D}_S = e^S \mathcal{D} e^{-S}$  [4].

Recall that, in the classical case, by a conserved and traceless energy-momentum tensor we mean that the right-hand sides of (2.1a) and (2.1c) vanish on the equations of motion. The equivalent statement in the ERG treatment is that the right-hand sides of the equations are ‘redundant’—with a redundant field defined such that it is generated by quasi-local field redefinition [15, 16]. (The term ‘inessential’ is also used in the literature.)

We are now ready to attempt to solve (2.1a), (2.1b) and (2.1c) in the ERG representation. The first step is to split the energy-momentum tensor up into a ‘classical’ piece and ‘quantum’ piece:

$$T_{\alpha\beta} = t_{\alpha\beta} + Q_{\alpha\beta}. \quad (2.96)$$

Conservation of the energy-momentum tensor implies:

$$\partial_\alpha T_{\alpha\beta} = e^S \frac{\delta}{\delta\varphi} \cdot K \times \partial_\beta \left( R \cdot \frac{\delta}{\delta\varphi} + K^{-1} \cdot \varphi \right) e^{-S}. \quad (2.97)$$

Neglecting the (divergent) vacuum term, (2.97) decomposes into (cf. (2.80))

$$\partial_\alpha t_{\alpha\beta} = -\partial_\beta \varphi \times \frac{\delta S}{\delta\varphi}, \quad (2.98a)$$

$$\partial_\alpha Q_{\alpha\beta} e^{-S} = \left\{ \frac{\delta}{\delta\varphi} \cdot K \times \partial_\beta R \cdot \frac{\delta}{\delta\varphi} - \partial_\beta \varphi \cdot K^{-1} \cdot (K \times \mathbb{1} - \mathbb{1} \times K) \cdot \frac{\delta}{\delta\varphi} \right\} e^{-S}. \quad (2.98b)$$

Since the classical part has been treated in section II B, we focus on the quantum contribution, the first term of which can be readily re-expressed by exploiting (2.81) and (2.84a):

$$\partial_\beta \varphi \cdot K^{-1} \cdot (K \times \mathbb{1} - \mathbb{1} \times K) \cdot \frac{\delta}{\delta\varphi} = \partial_\alpha \left( \partial_\beta \varphi \cdot K^{-1} \rtimes \mathcal{K}_\alpha \ltimes \frac{\delta}{\delta\varphi} \right). \quad (2.99)$$

To process the double derivative term, we again exploit (2.81) to pull out a total derivative piece:

$$\begin{aligned} \frac{\delta}{\delta\varphi} \cdot K \times \partial_\beta R \cdot \frac{\delta}{\delta\varphi} &= \frac{1}{2} \partial_\alpha \left( \delta_{\alpha\beta} \frac{\delta}{\delta\varphi} \cdot K \times R \cdot \frac{\delta}{\delta\varphi} \right. \\ &\quad \left. - \frac{\delta}{\delta\varphi} \cdot K \rtimes \mathcal{R}_\alpha \ltimes \partial_\beta \frac{\delta}{\delta\varphi} - \partial_\beta \frac{\delta}{\delta\varphi} \rtimes \mathcal{K}_\alpha \ltimes R \cdot \frac{\delta}{\delta\varphi} \right), \end{aligned} \quad (2.100)$$

where  $\mathcal{R}_\alpha$  is to  $R$  what  $\mathcal{K}_\alpha$  is to  $K$  is what  $\mathcal{F}_\alpha$  is to  $F$ . Equality follows straightforwardly from expanding out the right-hand side using (2.81) and noting that  $K \cdot R = R \cdot K$  (which is particularly obvious in momentum space). Thus we can construct the following contribution to the energy-momentum tensor:

$$\begin{aligned} \overline{Q}_{\alpha\beta} e^{-S} &= - \left( \partial_\beta \varphi \cdot K^{-1} \rtimes \mathcal{K}_\alpha \ltimes \frac{\delta}{\delta\varphi} - \frac{1}{2} \delta_{\alpha\beta} \frac{\delta}{\delta\varphi} \cdot K \times R \cdot \frac{\delta}{\delta\varphi} \right. \\ &\quad \left. + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot K \rtimes \mathcal{R}_\alpha \ltimes \partial_\beta \frac{\delta}{\delta\varphi} + \frac{1}{2} \partial_\beta \frac{\delta}{\delta\varphi} \rtimes \mathcal{K}_\alpha \ltimes R \cdot \frac{\delta}{\delta\varphi} \right) e^{-S}. \end{aligned} \quad (2.101)$$

To form the symmetric energy-momentum tensor, according to the recipe which ultimately produces (2.14), we use (2.87b) and (2.87c) to re-express in terms of symmetric pieces plus total derivative terms:

$$\Psi \rtimes \mathcal{F}_\alpha \ltimes \partial_\beta \Phi = -\partial_\alpha \left( \Psi \rtimes \mathcal{F} \ltimes \partial_\beta \Phi \right) + 2\Psi \rtimes \mathcal{F} \ltimes \partial_\alpha \partial_\beta \Phi, \quad (2.102a)$$

$$\partial_\beta \Psi \rtimes \mathcal{F}_\alpha \ltimes \Phi = \partial_\alpha \left( \partial_\beta \Psi \rtimes \mathcal{F} \ltimes \Phi \right) - 2\partial_\alpha \partial_\beta \Psi \rtimes \mathcal{F} \ltimes \Phi. \quad (2.102b)$$

Utilizing the manifest symmetry under  $\alpha \leftrightarrow \beta$  of the final terms allows us to decompose  $\overline{Q}_{\alpha\beta}$  along the lines of (2.6), with:

$$\begin{aligned} \overline{Q}_{\alpha\beta}^{\text{sym}} e^{-S} = & \left( 2\partial_\alpha \partial_\beta \varphi \cdot K^{-1} \rtimes \mathcal{K} \ltimes \frac{\delta}{\delta\varphi} + \frac{1}{2} \delta_{\alpha\beta} \frac{\delta}{\delta\varphi} \cdot K \times R \cdot \frac{\delta}{\delta\varphi} \right. \\ & \left. - \frac{\delta}{\delta\varphi} \cdot K \rtimes \mathcal{R} \ltimes \partial_\alpha \partial_\beta \frac{\delta}{\delta\varphi} + \partial_\alpha \partial_\beta \frac{\delta}{\delta\varphi} \rtimes \mathcal{K} \ltimes R \cdot \frac{\delta}{\delta\varphi} \right) e^{-S} \end{aligned} \quad (2.103)$$

and

$$F_{\lambda\alpha\beta} e^{-S} = -\delta_{\lambda\alpha} \left( \partial_\beta \varphi \cdot K^{-1} \rtimes \mathcal{K} \ltimes \frac{\delta}{\delta\varphi} - \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot K \rtimes \mathcal{R} \ltimes \partial_\beta \frac{\delta}{\delta\varphi} + \frac{1}{2} \partial_\beta \frac{\delta}{\delta\varphi} \rtimes \mathcal{K} \ltimes R \cdot \frac{\delta}{\delta\varphi} \right) e^{-S}. \quad (2.104)$$

As before, the next step is to construct the trace using (2.15) and, after adding the classical piece, to compare with the ERG version of (2.1c). The first step gives:

$$\begin{aligned} Q_{\alpha\alpha} = & \partial_\lambda \partial_\tau I_{\tau\lambda} + e^S \left\{ -\partial_\lambda \varphi \cdot K^{-1} \rtimes [\mathcal{K}_\lambda + (1-d)\partial_\lambda \mathcal{K}] \ltimes \frac{\delta}{\delta\varphi} + \frac{d}{2} \frac{\delta}{\delta\varphi} \cdot K \times R \cdot \frac{\delta}{\delta\varphi} \right. \\ & \left. - \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot K \rtimes [\mathcal{R}_\lambda + (d-1)\partial_\lambda \mathcal{R}] \ltimes \partial_\lambda \frac{\delta}{\delta\varphi} - \frac{1}{2} \partial_\lambda \frac{\delta}{\delta\varphi} \rtimes [\mathcal{K}_\lambda + (1-d)\partial_\lambda \mathcal{K}] \ltimes R \cdot \frac{\delta}{\delta\varphi} \right\} e^{-S} \end{aligned} \quad (2.105)$$

where, recalling (2.17), we have split  $H_{\tau\lambda}$  into classical and quantum pieces:

$$H_{\tau\lambda} = h_{\tau\lambda} + I_{\tau\lambda}. \quad (2.106)$$

Again, the strategy is to simplify by absorbing  $\mathcal{O}(\partial^2)$  pieces into the first term on the right-hand side of (2.105). To proceed we exploit (2.87c) to re-express

$$\Psi \rtimes (\mathcal{F}_\lambda + (1-d)\partial_\lambda \mathcal{F}) \ltimes \Phi = -(d-2)\Psi \rtimes \partial_\lambda \mathcal{F} \ltimes \Phi - 2\partial_\lambda \Psi \rtimes \mathcal{F} \ltimes \Phi, \quad (2.107)$$

and then utilize (2.84a) and (2.91):

$$\partial_\lambda \Psi \rtimes \mathcal{F} \ltimes \Phi = -\delta_0 \partial_\lambda (\Psi \cdot \{F', \mathbb{1}\} \cdot \Phi) - \partial_\lambda \Psi \cdot \{F', \mathbb{1}\} \cdot \Phi + \mathcal{O}(\partial^2). \quad (2.108)$$

Recalling (2.75) which, with our notation, amounts to  $G/2 \equiv K'$  gives, for some  $I_{\tau\lambda}^{(1)}$ :

$$\begin{aligned} -\partial_\lambda \varphi \cdot K^{-1} \rtimes (\mathcal{K}_\lambda + (1-d)\partial_\lambda \mathcal{K}) \ltimes \frac{\delta}{\delta\varphi} = & \partial_\lambda \partial_\tau I_{\tau\lambda}^{(1)} \\ & + \frac{\delta_0}{2} \partial_\lambda \left( \partial_\lambda \varphi \cdot K^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} \right) - \frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} \end{aligned} \quad (2.109)$$

where, in the final term on the right-hand side, we have used (2.77) to set  $\partial^2 K^{-1} = -\mathcal{G}^{-1}$ .



To treat the double functional derivative terms in (2.105) note that, for quasi-local  $A_1$  and  $A_2$

$$\frac{\delta}{\delta\varphi} \cdot A_1 \times A_2 \cdot \partial_\lambda \frac{\delta}{\delta\varphi} = O(\partial).$$

This can be seen by integrating: the integrand of the left-hand side is odd and so vanishes. This result allows us to finesse a number of terms at  $O(\partial^2)$ :

$$\frac{1}{2} \partial_\lambda \frac{\delta}{\delta\varphi} \times (\mathcal{K}_\lambda - (d-1)\partial_\lambda \mathcal{K}) \times R \cdot \frac{\delta}{\delta\varphi} = -\frac{1}{2} \frac{\delta}{\delta\varphi} \times G \cdot R \cdot \partial^2 \frac{\delta}{\delta\varphi} - \partial_\lambda \partial_\tau I_{\tau\lambda}^{(2)}. \quad (2.110)$$

Noting from (2.94) that

$$\frac{dR}{dp^2} = \frac{1}{p^2} \left( (\eta/2 - 1)R - \frac{G(p^2)}{2K(p^2)} \right) + \frac{G(p^2)}{2K(p^2)} R, \quad (2.111)$$

we can process the other double derivative term according to:

$$\begin{aligned} & \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot K \times (\mathcal{R}_\lambda + (d-1)\partial_\lambda \mathcal{R}) \times \partial_\lambda \frac{\delta}{\delta\varphi} e^{-S} \\ &= \left[ (1 - \eta/2) \frac{\delta}{\delta\varphi} \cdot K \times R \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \times \frac{\delta}{\delta\varphi} + \frac{1}{2} \frac{\delta}{\delta\varphi} \times G \cdot R \cdot \partial^2 \frac{\delta}{\delta\varphi} - \partial_\lambda \partial_\tau I_{\tau\lambda}^{(3)} \right] e^{-S}. \end{aligned} \quad (2.112)$$

Substituting (2.109), (2.110) and (2.112) into (2.105) gives:

$$\begin{aligned} Q_{\alpha\alpha} e^{-S} = & - \left[ \frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \times \frac{\delta}{\delta\varphi} - \delta \frac{\delta}{\delta\varphi} \cdot K \times R \cdot \frac{\delta}{\delta\varphi} \right. \\ & \left. - \frac{\delta_0}{2} \partial_\lambda \left( \partial_\lambda \varphi \cdot K^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} \right) - \partial_\lambda \partial_\tau (I + I^{(1)} + I^{(2)} + I^{(3)})_{\tau\lambda} \right] e^{-S}. \end{aligned} \quad (2.113)$$

For the full quantum field theoretic case, the trace of the energy-momentum tensor is given by:

$$T_{\alpha\alpha} = \delta e^S \frac{\delta}{\delta\varphi} \cdot K \times \left( R \cdot \frac{\delta}{\delta\varphi} + K^{-1} \cdot \varphi \right) e^{-S}. \quad (2.114)$$

Extracting the ‘quantum’ part, taking

$$\partial_\lambda \partial_\tau I_{\tau\lambda}^{(4)} = \delta e^S \partial_\lambda \left[ \frac{1}{2} \partial_\lambda \varphi \cdot K^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} + \varphi \cdot K^{-1} \times \mathcal{K}_\lambda \times \frac{\delta}{\delta\varphi} \right] e^{-S} \quad (2.115)$$

and comparing with (2.113) gives the constraint:

$$\begin{aligned} \partial_\lambda \partial_\tau \tilde{I}_{\tau\lambda} e^{-S} + \dots = & \left[ \frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} \right. \\ & \left. + \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \times \frac{\delta}{\delta\varphi} + \frac{\eta}{4} \partial_\lambda \left( \partial_\lambda \varphi \cdot K^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta\varphi} \right) \right] e^{-S} + \dots, \end{aligned} \quad (2.116)$$

where the ellipsis on each side represents the omitted classical terms and

$$\partial_\lambda \partial_\tau \tilde{I}_{\tau\lambda} = \partial_\lambda \partial_\tau (I + I^{(1)} + I^{(2)} + I^{(3)} + I^{(4)})_{\tau\lambda}. \quad (2.117)$$

Finally, we add into (2.116) the classical contributions from (2.45) to give the full result:

$$\begin{aligned} \partial_\lambda \partial_\tau \tilde{H}_{\tau\lambda} = -e^S & \left\{ d\hat{L} - \sum_{i=1}^{\infty} [D_{\underline{a}_1^i}, x \cdot \partial] \varphi \times \frac{\partial \hat{L}}{\partial (D_{\underline{a}_1^i} \varphi)} \right. \\ & + \delta \varphi \times \frac{\delta}{\delta \varphi} + \frac{1}{2} \varphi \cdot \mathcal{G}^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta \varphi} + \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot G \times \frac{\delta}{\delta \varphi} \\ & + \partial_\lambda \left( \frac{1}{2} (\delta_{\omega\lambda} \delta_{\rho\sigma} - 2\delta_{\omega\rho} \delta_{\sigma\lambda}) \sum_{i=2}^{\infty} [[D_{\underline{a}_1^i}, x_\sigma], x_\rho \partial_\omega] \varphi \times \frac{\partial \hat{L}}{\partial (D_{\underline{a}_1^i} \varphi)} \right. \\ & \left. \left. + \frac{\eta}{4} \partial_\lambda \varphi \cdot K^{-1} \cdot \{G, \mathbb{1}\} \cdot \frac{\delta}{\delta \varphi} \right) \right\} e^{-S} \quad (2.118) \end{aligned}$$

where, recalling (2.106), (2.44) and (2.117),

$$H_{\tau\lambda} = \tilde{H}_{\tau\lambda} + (I - \tilde{I})_{\tau\lambda} + (h - \tilde{h})_{\tau\lambda}. \quad (2.119)$$

We can check consistency of (2.118) just as we did in the classical case: first, we integrate the equation as it stands. The total derivative terms vanish; the first three terms on the right-hand side have already been dealt with in the reduction of (2.45) to (2.47) and it is easy to see that they combine with the remaining terms combine to give the ERG equation (2.79a).

Returning to (2.118), now we multiply by  $2x_\mu$  and then integrate. In this case, all four ‘classical’ terms have been processed in the reduction of (2.45) to (2.51). The various pieces combine, straightforwardly, to give the special conformal partner of ERG equation, (2.79b).

### III. CONCLUSION

The starting point for the main analysis of this paper is the defining equations for the energy-momentum tensor (2.1a), (2.1b) and (2.1c), written in an arbitrary representation of the conformal algebra. These three equations respectively encode translation, rotation and dilatation invariance. Supposing that the first constraint can be solved, it is possible to solve the second also to arrive at a conserved, symmetric tensor along the lines of the Belinfante tensor. However, additionally imposing dilatation invariance produces a constraint equation (2.19); solutions of this equation, should they exist, provide the requisite improvement

to the energy-momentum tensor, while self consistently determining the action and scaling dimension of the fundamental field.

This scheme for constructing the energy-momentum tensor is explored in two concrete representations of the conformal algebra. The treatment of classical theories has a rather standard feel, with the novelty—such as there is one—arising from allowing the Lagrangian to contain terms with an arbitrary number of derivatives. The motivation for this is not that we are interested in classical theories of this type, per se, but rather that such terms necessarily arise in the full ERG treatment which follows. The classical analysis also provides a relatively simple setting in which to directly see that the constraint equation (2.19) directly encodes both dilatation and special conformal invariance of the action.

For a classical CFT, the energy-momentum tensor can be reconstructed as follows: a conserved, symmetric tensor can be built from (2.35) and (2.40b) using the recipe (2.14) but excluding the final term. The latter improvement is determined by substituting the solution to the constraint equation (2.45) into (2.44) and then substituting the result into either (2.18a) or (2.18b), as appropriate.

The ERG analysis is facilitated by the new notation introduced in (2.82). This makes it relatively straightforward to solve the conservation equation (2.1a) which, in the ERG representation, translates to (2.97). The solution separates into classical and quantum pieces, with the former dealt with already and the latter given by (2.101). The new notation swiftly enables the extraction of  $F_{\lambda\alpha\beta}$  in (2.104); at this stage a conserved, symmetric tensor could be readily constructed, again by using (2.14) modulo the last term. The final part of the analysis involves the improvement of the energy-momentum tensor. The goal of obtaining the ERG representation of the constraint equation (2.19) is achieved in the ‘conformal fixed-point equation’, (2.118)—the exploration of which is deferred to future work.

Just as the classical version of this equation encodes dilatation and special conformal invariance of the action, so (2.118) directly encodes the ERG equation (2.79a) and its special conformal partner (2.79b). Solutions to (2.118) yield CFTs and generate the improvement term of the energy-momentum tensor upon substitution of (2.119) into either (2.18a) or (2.18b).

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## Appendix A: ERG Conventions

Whereas, in this paper, we work with the full Wilsonian effective action,  $S$ , more usually the ERG equation is phrased in terms of  $\mathcal{S}$ , defined via:

$$S = \frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi + \mathcal{S}. \quad (\text{A1})$$

In terms of  $\mathcal{S}$ , the fixed-point ERG equation and its partner expressing special conformal invariance are:

$$\left\{ D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2}\frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} - \frac{\eta}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi \right\} e^{-\mathcal{S}} = 0, \quad (\text{A2a})$$

$$\left\{ K^{(\delta)}{}_{\mu}\varphi \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2}\frac{\delta}{\delta\varphi} \cdot G_{\mu} \cdot \frac{\delta}{\delta\varphi} - \frac{\eta}{2}\varphi \cdot \mathcal{G}^{-1}{}_{\mu} \cdot \varphi + \eta\varphi \cdot K^{-1} \cdot [X_{\mu}, K] \cdot \frac{\delta}{\delta\varphi} \right\} e^{-\mathcal{S}} = 0. \quad (\text{A2b})$$

We now show how to bring these equations into the form utilized in the rest of the paper. Starting with (A2a), observe that:

$$\begin{aligned} e^{-\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \left[ D^{(\delta)}\varphi \cdot \frac{\delta}{\delta\varphi}, e^{\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \right] &= -\frac{1}{2}\varphi \cdot (D^{(d-\delta)}\mathcal{G}^{-1} + \mathcal{G}^{-1}\overleftarrow{D}^{(d-\delta)}) \cdot \varphi \\ &= \frac{1}{2}\varphi \cdot (\eta\mathcal{G}^{-1} - \mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1}) \cdot \varphi \end{aligned} \quad (\text{A3})$$

and also:

$$e^{-\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \left[ \frac{1}{2}\frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi}, e^{\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \right] = \varphi \cdot \mathcal{G}^{-1} \cdot G \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1} \cdot \varphi \quad (\text{A4})$$

where, consistent with the rest of this paper, we have ignored a vacuum term on the right-hand side. It is thus apparent that (A2a) transforms into (2.79a).

To process (A2b), we first bring the final term into a simpler form. From (2.75) we see that that

$$\begin{aligned} \frac{\partial}{\partial x_{\alpha}}((x-y)_{\alpha}K) &= (d+x \cdot \partial_x - y \cdot \partial_x)K \\ &= (d+x \cdot \partial_x + y \cdot \partial_y)K = \partial^2 G. \end{aligned} \quad (\text{A5})$$

From this, we deduce that

$$[X_{\alpha}, K] = \partial_{\alpha}G, \quad (\text{A6})$$

and so

$$\varphi \cdot K^{-1} \cdot [X_\alpha, K] \cdot \frac{\delta}{\delta\varphi} = -\partial_\alpha \varphi \cdot K^{-1} \cdot G \cdot \frac{\delta}{\delta\varphi}. \quad (\text{A7})$$

Commuting this term with  $e^{\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi}$  leaves behind

$$-\partial_\alpha \varphi \cdot K^{-1} \cdot G \cdot \mathcal{G}^{-1} \cdot \varphi = 0, \quad (\text{A8})$$

with this result following since the integrand on the right-hand side is odd.

Continuing with (A2b) we exploit the following result for some  $U((x-y)^2)$  and some  $V((x-y)^2)$ . If we suppose that

$$D^{(\Delta)}U + U\overleftarrow{D}^{(\Delta)} = V. \quad (\text{A9})$$

then it follows that

$$K^{(\Delta)}_\mu U + U\overleftarrow{K}^{(\Delta)}_\mu = V \times X_\mu + X_\mu \times V. \quad (\text{A10})$$

Immediately, we see that

$$e^{-\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \left[ K^{(\delta)}_\mu \varphi \cdot \frac{\delta}{\delta\varphi}, e^{\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \right] = \frac{1}{2}\varphi \cdot (\eta \mathcal{G}^{-1} - \mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1})_\mu \cdot \varphi. \quad (\text{A11})$$

The second result we require is

$$e^{-\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \left[ \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G_\mu \cdot \frac{\delta}{\delta\varphi}, e^{\frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot \varphi} \right] = \varphi \cdot \mathcal{G}^{-1} \cdot G_\mu \cdot \frac{\delta}{\delta\varphi} + \frac{1}{2}\varphi \cdot \mathcal{G}^{-1} \cdot G_\mu \cdot \mathcal{G}^{-1} \cdot \varphi. \quad (\text{A12})$$

Next, we combine various terms:

$$\varphi \cdot (\mathcal{G}^{-1} \cdot G_\mu \cdot \mathcal{G}^{-1} - (\mathcal{G}^{-1} \cdot G \cdot \mathcal{G}^{-1})_\mu) \cdot \varphi = \varphi \cdot ([\mathcal{G}^{-1}, X_\mu] \cdot G \cdot \mathcal{G}^{-1} - \mathcal{G}^{-1} \cdot G \cdot [\mathcal{G}^{-1}, X_\mu]) \cdot \varphi = 0, \quad (\text{A13})$$

where we have exploited that each term is separately odd and so the integrals vanish. Putting everything together reproduces (2.79b).

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